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THE DEFECT-CORRECTION METHOD APPLIED TO SINGULARLY PERTURBED ELLIPTIC PROBLEMS

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Abstract— In this paper the defect–correction method has been applied to solve some singularly perturbed elliptic boundary value problems of convection-diffusion type. From the available literature from an earlier work, it was proved that the use of standard defect–correction technique allows one to improve the order of convergence of stable low order finite difference schemes. However, it is difficult to prove the uniform convergence for one dimensional problem and we do not know at present any theoretical result proving uniform convergence for general two dimensional elliptic problems. By an another approach proposed, it was found that a simplified analysis is used for one dimensional problems of convection-diffusion type which gives almost second order uniform convergence of the method. In this work we show numerically that it is possible to extend this parameter-uniform method and the simplified analysis to the case of a two dimensional elliptic boundary value problems.

Index Terms— Defect-correction, finite difference method, Shishkin mesh, singular perturbation, uniform convergence.
AMS classification: 65M06, 65M12, 65L70, 65L12.

1 Introduction

Singularly perturbed problems are characterized by the presence of a small parameter multiplying the highest order derivatives of the differential equation. The presence of the singular perturbation parameter results in the solution of these problems having a multi-scale character. That is, narrow regions (boundary layers) appear close to the boundary where the solution has large gradients. So, it is desirable to design special numerical methods which generate numerical approximations converging to the exact solution independently of the value of the singular perturbation parameter. These methods are called uniformly convergent methods or parameter uniform methods.

In [2] some parameter uniform fitted mesh methods were presented for problems of this kind, using exclusively a special class of meshes introduced by Shishkin. These meshes are simple to construct, are piecewise uniform and condense half the mesh points into the boundary layer regions. In the literature there are many first order finite difference schemes used to approximate efficiently a wide class of one and two dimensional singularly perturbed problems. Moreover, it is possible to apply some classical techniques to improve the uniform order of convergence in the one dimensional case. For the convection–diffusion elliptic problems given by

$$L_\varepsilon u \equiv \varepsilon \Delta u + a.\nabla u - bu = f,\ (x,y)\in\Omega=(0,1)^2,\ u=g \text{ on } \partial\Omega, \quad (1)$$

where the author proves almost second order uniform convergence in the maximum norm [6]. In that paper, the Richardson extrapolation technique was applied to the upwind operator defined on a special Shishkin mesh. It is well known that the Richardson extrapolation technique requires one to solve two discrete problems. The basic discrete problem associated with the original mesh and a second discrete problem associated with a new mesh composed of the mesh points of the original mesh and their midpoints. Note also that the analysis of the uniform convergence of this technique employs an asymptotic expansion of the error in powers of the discretization parameter.

This paper is motivated by our interest in high order parameter uniform finite difference schemes. Here, we consider the defect–correction method to achieve almost second order uniform convergence. This method was applied in [3] for one dimensional problems, where the upwind operator (stable) and the central difference operator (unstable) were combined; unfortunately, the analysis of the uniform convergence, proposed in that work, is complicated and it appears difficult to extend to two dimensional elliptic problems. In [4] a new basic stable scheme was proposed, simplifying the analysis of the uniform convergence in the case of one dimensional problem.

The paper is presented as follows. In section 2, we describe the new method and in section 3 we show that it can be extended to the two dimensional case. The numerical results suggest almost second order uniform convergence of the numerical method. Hereinafter, C denotes a positive constant independent of the singular perturbation parameter ε and the discretization parameter N.

2. DEFECT CORRECTION METHOD FOR ONE DIMENSIONAL PROBLEM

Here we consider boundary value problems for ordinary differential equation of convection–diffusion type given by

$$L_\varepsilon u \equiv \varepsilon u'' + au' - bu = f,\ x \in \Omega = (0,1),\ u(0) = u_0,\ u(1) = u_1, \quad (2)$$

where we assume that the singular perturbation parameter ε can take small positive values, $0 < \varepsilon \le 1$, the coefficients $a, b, f \in C^5(0,1)$ and $a \ge \alpha > 0$, $b \ge 0$, for all $x \in [0,1]$. The solution of (2) has a boundary layer at $x = 0$ having a width $O(\varepsilon \ln(1/\varepsilon))$; moreover (see [2]) its derivatives satisfy the bounds

$$|u^{(k)}(x)| \le C(1 + e^{-k} \exp(-\alpha x/\varepsilon)),\ 0 \le k \le 4. \quad (3)$$

To prove uniform convergence of the numerical scheme, it is convenient to use an appropriate decomposition of the exact solution u. Following [2], we write $u = v + w$, where v and w are the regular and singular components of the exact solution respectively. They are the solutions of the following boundary value problems:

$$L_\varepsilon v = f,\ v(0) = v^*(0),\ v(1) = u(1),$$
$$L_\varepsilon w = 0,\ w(0) = u(0) - v^*(0),\ w(1) = 0,$$

where $v^*(0)$ is taken so that

$$|v^{(j)}(x;\varepsilon)| \le C,\ 0 \le j \le 3,\ \ \varepsilon|v^{(4)}(x;\varepsilon)| \le C, \quad (4)$$

$$|w(x;\varepsilon)| \le C\exp(-\alpha x/\varepsilon),\ \ |w^{(j)}(x;\varepsilon)| \le C\varepsilon^{-j},\ 1 \le j \le 4 \quad (5)$$

To approximate the solution of (2), we consider a finite difference scheme defined on a Shishkin mesh. Let N be the discretization parameter; then the Shishkin mesh is defined by using the transition parameter

$$\sigma = \min\{1/2,\ (3/\alpha)\,\varepsilon \ln(N)\} \quad (6)$$

and dividing uniformly each one of the subdomains $[0,\sigma]$, $[\sigma,1]$ into $N/2$ intervals. Then, the set of mesh points $\overline{\Omega}_\varepsilon^N$ is given by

$$x_j = \begin{cases} jh, & 0 \le j \le N/2 \\ \sigma + (j - N/2)H, & N/2 \le j \le N, \end{cases} \quad (7)$$

where $h=2\sigma/N$ is the fine mesh step and $H = 2(1-\sigma)/N$ is the coarse mesh step. Below, we denote by $h_{j+1} = x_{j+1} - x_j$ for $j = 0, \ldots, N-1$, and $\bar{h}_j = (h_j + h_{j+1})/2$ for $j = 1, \ldots, N-1$. On this piecewise uniform mesh, we consider the following hybrid three point finite difference scheme:

$$L^N U_J \equiv r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = f_j,\ 0 < j < N,\ U_0 = u_0,\ U_N = u_1 \quad (8)$$

where the discrete operator L^N is defined by

$$L^N \equiv \begin{cases} L_{cd*}^N, & if\ 1 \le j < N/2 \\ L_{cd*}^N, & if\ N/2 \le j < N\ and\ \|a\| N^{-1} < \varepsilon \\ L_{up}^N, & if\ N/2 \le j < N\ and\ \|a\| N^{-1} \ge \varepsilon \end{cases}$$

The central difference operator L_{cd*}^N (slightly modi fied at the transition point) and the upwind operator L_{up}^N are given by

$$L_{cd*}^N U_j \equiv \varepsilon\delta^2 U_j + a_j D^{\pm} U_j - b_j U_j = f_j,$$
$$L_{up}^N U_j \equiv \varepsilon\delta^2 U_j + a_j D^{+} U_j - b_j U_j = f_j,$$

where

$$\delta^2 U_j = \frac{1}{\bar{h}_j}\left(\frac{U_{j+1} - U_j}{h_{j+1}} - \frac{U_j - U_{j-1}}{h_j}\right),$$

$$D^+ U_j = \frac{U_{j+1} - U_j}{h_{j+1}},\ \ D^- U_j = \frac{U_j - U_{j-1}}{h_{j+1}},$$

$$D^{\pm} U_j = \frac{h_j}{h_j + h_{j+1}} D^+ U_j + \frac{h_{j+1}}{h_j + h_{j+1}} D^- U_j.$$

Theorem 2.1 [4] The error associated with the hybrid scheme (8) satisfies

$$|u(x_j) - U_j| \le \begin{cases} CN^{-1}, & if\ \|a\| N^{-1} \ge \varepsilon, \\ CN^{-2}(\ln N)^3, & if\ \|a\| N^{-1} < \varepsilon, \end{cases} \quad (9)$$

and hence it is a first order uniformly convergent method.

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To improve this order of uniform convergence, we apply the defect–correction method. Note that the use of the hybrid scheme (8) instead of the upwind scheme allows one prove almost second order in the case $\|a\| N^{-1} < \varepsilon$, and therefore in this case it is not necessary to modify the numerical solution given by scheme (8). Otherwise, it is necessary to correct the numerical solution in an appropriate way to obtain second order convergence. In [4] the following approximation to the solution of problem (2) was proposed

$$\begin{cases} \hat{U} = U + \vartheta, & if \ \|a\| N^{-1} \geq \varepsilon, \\ \hat{U} = U, & otherwise. \end{cases} \tag{10}$$

Here ϑ is the solution of the discrete problem

$$L^N \vartheta = f - L^N_{cd} U \ \text{ in } \ \Omega, \ \ \vartheta = 0 \ \text{ in } \ \partial\Omega, \tag{11}$$

where L^N_{cd} is the classical central difference operator, which is defined by

$$L^N_{cd*} U_j \equiv \begin{cases} L^N_{cd*} Z_j, & if \ \ j \neq N/2, \\ \varepsilon \delta^2 Z_j + a_j D^0 Z_j - b_j Z_j = f_j, & if \ \ j = N/2, \end{cases}$$

In [4] it was showed that if $D^{\pm}$ is used instead of D^0 at the transition point to find the corrected solution, then the resulting numerical method is not parameter-uniformly convergent. The numerical solution of scheme (10) satisfies the following convergence result.

Theorem 2.2. [4] The error associated with the defect-correction scheme (10) satisfies
$\|u - \hat{U}\| \leq C N^{-2} (\ln N)^3$,
where u is the solution of the problem(2) and $\hat{U}$ is the numerical solution given in (10). Hence, the new method is almost second order uniformly convergent.

To confirm the theoretical result of convergence proved in Theorem 2.2, we consider the following test problem:

$$\varepsilon u'' + u' = -e^x (1+\varepsilon), \ \ x \in (0,1), \ u(0)=1, \ u(1)=1-e, \tag{12}$$

whose exact solution is
$u(x) = [e^{-(x/\varepsilon)} - e^{-(1/\varepsilon)}] / (1 - e^{-(1/\varepsilon)}) + 1 - e^x$.

Then, for any value of N, the maximum pointwise errors $E_{\varepsilon,N}$ and the ε–uniform errors are calculated by $E_{\varepsilon,N} = \max_i |u(x_i) - \hat{U}|$, $E_N = \max_\varepsilon E_{\varepsilon,N}$ respectively, where $u(x_i)$ is the exact solution of (12) and $\hat{U}$ is the numerical solution of the finite difference scheme (10). From these values the orders of convergence $p_{\varepsilon,N}$ and the order of ε–uniform convergence p_N are calculated using $p_{\varepsilon,N} = \log(E_{\varepsilon,N} / E_{\varepsilon,2N}) / \log 2$ and
$p_N = \log(E_N/E_{2N})/\log 2$.

Table 1 displays the ε–uniform maximum errors (E_N) and the ε–uniform orders of convergence (P_N) for the range of values ε = 1, 2^{-2}, 2^{-4}, . . ., 2^{-30}, for problem (12). From these results we see that almost second order uniform convergence in the maximum norm, which is in agreement with the bound in Theorem 2.2.

Table 1: Numerical results for problem (12)

	N=32	N=64	N=128	N=256	N=512	N=1024
E_N	1.641 e-2	5.483 e-3	1.781 e-3	5.662 e-4	1.758 e-4	5.353e-5
P_N	1.581	1.622	1.653	1.688	1.715	1.739

3. DEFECT CORRECTION METHOD FOR TWO DIMENSIONAL PROBLEMS

The aim of this section is to show that the ideas developed for the one dimensional case can be extended to elliptic boundary value problems given by

$$L_\varepsilon u \equiv \varepsilon \Delta u + a.\nabla u - bu = f, \ (x,y) \in \Omega = (0,1)^2, \ u = g \ \text{ on } \partial\Omega \tag{13}$$

where the coefficients of the differential equation a_1, a_2, b and f are sufficiently smooth and they satisfy $a = (a_1(x), a_2(y)) \geq (\alpha_1, \alpha_2) > (0, 0)$, $b(x, y) \geq 0$, in Ω. Further, we suppose that sufficient compatibility conditions hold in order that $u \in C^{4,\alpha}(\Omega)$ (space of Hölder continuous functions whose derivatives up to fourth order exist and they are Hölder continuous). To achieve this regularity, compatibility conditions up to second level are necessary (see [5]).

Using an appropriate change of variable and a classical analysis (see [7]), we can deduce the following crude bounds for the exact solution and its partial derivatives in the maximum norm
$\|u^{(i,j)}\| \leq C \varepsilon^{-(i+j)}$, $0 \leq i + j \leq 4$.

Also, it is well-known (see [2]) that the exact solution of (13) has two regular boundary layers near the sides $x = 0$ and $y = 0$ of width $O(\varepsilon \ln 1/\varepsilon)$.

To approximate the solution of (13), again we consider a piecewise uniform mesh $\overline{\Omega}^N = \{(x_i, y_j)\}_{i,j=0}^N$, which is the tensor product of the corresponding Shishkin mesh considered for the one dimensional problem. Then, the mesh points are

$$x_i = \begin{cases} ih, & 0 \le i \le N/2 \\ x_{N/2} + (i - N/2)H, & N/2 < i \le N, \end{cases}$$

$$y_j = \begin{cases} jk, & 0 \le j \le N/2 \\ y_{N/2} + (j - N/2)K, & N/2 < j \le N, \end{cases}$$

where $h = 2\sigma_x/N$, $k = 2\sigma_y/N$, $H = 2(1-\sigma_x)/N$, $K = 2(1-\sigma_y)/N$ and the transition parameters are defined by

$\sigma_x = \min\{1/2, (3/\alpha_1)\, \varepsilon \ln(N)\}$,
$\sigma_y = \min\{1/2, (3/\alpha_2)\, \varepsilon \ln(N)\}$.

Let us denote the local step sizes by
$h_j = x_j - x_{j-1}$, $k_j = y_j - y_{j-1}$, $j = 1,2,\dots,N$,
$\overline{h}_j = (h_j + h_{j+1})/2$, $\overline{k}_j = (k_j + k_{j+1})/2$, $j = 1,2,\dots,N-1$.

On this mesh we define the following hybrid scheme

$$\begin{cases} L^N U_{i,j} \equiv \varepsilon(\delta_x^2 + \delta_y^2)U_{i,j} + a_1(x_i)D_x U_{i,j} + a_2(y_j)D_y U_{i,j} \\ \qquad -b(x_i, y_j)U_{i,j} = f(x_i, y_j) \ \ in\ \Omega^N, \quad (14) \\ U_{i,j} = g(x_i, y_j) \ \ on\ \partial\Omega^N, \end{cases}$$

where

$$\delta_x^2 U_{i,j} = \frac{1}{\overline{h}_i}(D_x^+ U_{i,j} - D_x^- U_{i,j}),$$

$$\delta_y^2 U_{i,j} = \frac{1}{\overline{k}_j}(D_y^+ U_{i,j} - D_y^- U_{i,j}),$$

$$D_x U_{i,j} \equiv \begin{cases} D_x^+ U_{i,j}, & if\ \varepsilon \le \|a_1\| N^{-1} \\ D_x^{\pm} U_{i,j}, & if\ \|a_1\| N^{-1} < \varepsilon\ or\ x_i < \sigma_x, \end{cases}$$

$$D_y U_{i,j} \equiv \begin{cases} D_y^+ U_{i,j}, & if\ \varepsilon \le \|a_2\| N^{-1} \\ D_y^{\pm} U_{i,j}, & if\ \|a_2\| N^{-1} < \varepsilon\ or\ y_j < \sigma_y, \end{cases}$$

$$D_x^{\pm} U_{i,j} = \frac{h_i}{h_i + h_{i+1}} D_x^+ U_{i,j} + \frac{h_{i+1}}{h_i + h_{i+1}} D_x^- U_{i,j},$$

$$D_y^{\pm} U_{i,j} = \frac{k_j}{k_j + k_{j+1}} D_y^+ U_{i,j} + \frac{k_{j+1}}{k_j + k_{j+1}} D_y^- U_{i,j},$$

$$D_x^+ U_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h_{i+1}}, \quad D_x^- U_{i,j} = \frac{U_{i,j} - U_{i-1,j}}{h_i},$$

$$D_y^+ U_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{k_{j+1}}, \quad D_y^- U_{i,j} = \frac{U_{i,j} - U_{i,j-1}}{k_j}.$$

Following (8), it is not difficult to prove that the error associated to the hybrid scheme (14) satisfies $\|u - U\| \le CN^{-1}$, proving first order uniform convergence. To improve this order, similar to the one dimensional case we propose the following defect–correction scheme

$$\begin{cases} \hat{U} = U + \vartheta, & if\ \max\{\|a_1\|, |a_1\|\}N^{-1} \ge \varepsilon, \\ \hat{U} = U, & otherwise. \end{cases} \qquad (15)$$

where now ϑ is the solution of the discrete problem

$$L^N \vartheta = f - L_{cd}^N U \ \ in\ \Omega, \ \ \vartheta = 0 \ \ on\ \partial\Omega, \qquad (16)$$

with

$$L_{cd}^N U_{i,j} \equiv \varepsilon(\delta_x^2 + \delta_y^2)U_{i,j} + a_1(x_i)D_x^0 U_{i,j} + a_2(y_j)D_y^0 U_{i,j} \\ -b(x_i, y_j)U_{i,j},$$

$$D_x^0 U_{i,j} = \frac{U_{i+1,j} - U_{i-1,j}}{h_{i+1} + h_i}, \quad D_y^0 U_{i,j} = \frac{U_{i,j+1} - U_{i,j-1}}{k_{j+1} + k_j}.$$

To illustrate the uniform convergence of this defect–correction method numerically, we show the numerical results obtained for two test problems, whose exact solution in both cases is unknown. To estimate the pointwise errors of the solution $\{\hat{U}^N\}$, we use a variant of the double mesh principle [1]; then, we calculate a new approximation $\{\hat{V}^{2N}\}$ on the mesh $\tilde{\Omega}^{2N} = \{\tilde{x}_i, \tilde{y}_j\}$ that uses the mesh points of the original Shishkin mesh and their midpoints,

$$\tilde{x}_{2i} = x_i, \ \ \tilde{y}_{2j} = y_j, \ i, j = 0,1,\dots N, \qquad (17)$$

$$\tilde{x}_{2i+1} = (x_i + x_{i+1})/2, \ \tilde{y}_{2j+1} = (y_j + y_{j+1})/2, \ \ i, j = 0,1,\dots N-1. \qquad (18)$$

In this way, we compare both numerical solutions at the mesh points of the coarse mesh, i.e., we calculate $D_{\varepsilon,N}(x_i, y_j) = |\hat{U}_{i,j}^N - \hat{V}_{2i,2j}^{2N}|$. For each fixed value of ε, the maximum errors and the numerical orders of convergence are computed by $D_{\varepsilon,N} = \max_{i,j} D_{\varepsilon,N}(x_i, y_j)$ and $q_{\varepsilon,N} = \log(D_{\varepsilon,N}/ D_{\varepsilon,2N})/ \log 2$ respectively. From these values we calculate the ε-uniform errors and the ε-uniform orders of convergence $D_N = \max_\varepsilon D_{\varepsilon,N}$ and $q_N = \log(D_N/ D_{2N})/ \log 2$.

The first test problem is given by

$$\varepsilon\Delta u+(1+x)u_x +(2+y)u_y - u = (1-x)+(1-y) \text{ in } \Omega = (0,1)^2, \quad u = 1 \text{ on } \partial\Omega. \qquad (19)$$

The data of this problem do not satisfy sufficient compatibility conditions in order that the exact solution has the required regularity (see [5]). Concretely, we only have compatibility conditions of level zero, that is, the boundary conditions are continuous in the four corners of the unit square. Table 2 displays the numerical results for the same range of values of ε as before, giving in the first row the ε–uniform errors and in the second one the corresponding uniform orders of convergence. From this table we clearly see that we do not have a second order rate of uniform convergence.

Table 2: Numerical results for problem (19)

	N=16	N=32	N=64	N=128	N=256
D_N	0.310e-1	0.127e-1	0.485e-2	0.329e-2	0.228e-2
q_N	1.288	1.388	0.561	0.528	

The second test problem is given by

$$\varepsilon\Delta u+(1+x)u_x +(2+y)u_y = (1-x)+(1-y) \text{ in } \Omega = (0,1)^2, \quad u = 0 \text{ on } \partial\Omega. \qquad (20)$$

Now, we again have compatibility conditions of level zero at three corners of the unit square, and we can check that in the inflow corner (1, 1) the first level of compatibility is satisfied. Table 3 displays that the scheme (15) is amost second order uniformly convergent. Therefore, we can conjecture that the defect correction technique improves the order of convergence of the basic hybrid scheme when a sufficient level of compatibility holds.

Table 3: Numerical results for problem (20)

	N=16	N=32	N=64	N=128	N=256
D_N	0.225e-1	0.827e-2	0.283e-2	0.925e-3	0.298e-3
q_N	1.445	1.547	1.612	1.633	

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